

Scattering Relativity in Quantum Mechanics

Richard Shurtleff *

August 15, 2011

Abstract

Transforming from one reference frame to another yields an equivalent physical description. If quantum fields are transformed one way and quantum states transformed a different way then the physics changes. We show how to use the resulting changed physical description to obtain the equations of motion of charged, massive particles in electromagnetic and gravitational fields. The derivation is based entirely on special relativity and quantum mechanics.

Keywords: Special Relativity, Quantum Fields, Lorentz Force Law, Geodesics

PACS numbers: 03.70.+k, 03.65.Sq, 11.30.Cp,

1 Introduction

The principle of relativity and the principles of quantum mechanics can be combined to determine fundamental aspects of the quantum theory of fields.[1] The work here continues that point of view with another way of using the symmetry transformations of special relativity.

Briefly stated, the principle of relativity maintains that transforming the vectors, spinors, and tensors of a given physical situation to a new reference frame results in an equivalent description of the physical situation. The key is that *all* physical quantities are to be transformed.

*affiliation and mailing address: Department of Applied Mathematics and Sciences, Wentworth Institute of Technology, 550 Huntington Avenue, Boston, MA, USA, ZIP 02115, telephone number: (617) 989-4338, fax number: (617) 989-4591 , e-mail address: shurtleffr@wit.edu

Obversely, if quantities are transformed, some one way and others differently, then the physical description is not equivalent to the original. This can be useful. The process is ‘scattering’, separating spacetime transformations so that different quantities go off in different directions.

We work in a flat spacetime (x, η) with coordinates x^α and metric $\eta_{\alpha\beta}$, where $\alpha, \beta \in \{1, 2, 3, 4\}$ with $4 = t$ being time. When needed we use $\eta = \text{diag}\{+1, +1, +1, -1\}$.

The particle fields and states are for a single species of particles with mass m . Any particle momentum p is a four-vector that, by a Poincaré transformation, can be transformed to $k = \{0, 0, 0, m\}$ in the ‘rest frame’. Since spacetime scalar products are preserved under Poincaré transformations, we have

$$\eta_{\alpha\beta} p^\alpha p^\beta = -m^2, \quad (1 : \text{Mass}) \quad (1)$$

where repeated indices are summed.

We base our work on a derivation from the above mentioned quantum theory of fields.[1] The derivation of the quantum field transforms both the field and the states by a Poincaré transformation, which is a combination of rotations, boosts, and translations connected to the identity.

With scattering, a different transformation is applied to the field than the one applied to the states. The standard calculation is then followed closely. Among the results, the generalization puts an arbitrary tensor field $A(x)$ in the various phases. For momentum p , instead of phase $\Theta = p \cdot x = \eta_{\alpha\beta} p^\alpha x^\beta$, we get the phase $\Theta = p \cdot Ax = \eta_{\alpha\beta} p^\alpha A_\mu^\beta x^\mu$, where x^μ are spacetime coordinates.

Choosing to apply a method we can work with, we seek paths of extreme phase. Over a short interval δx , let the extreme phase change be $\delta\Theta = -m\delta\tau$, which defines τ . Calculus shows that extreme phase change requires that momentum p and $A\delta x$ are in the same direction. We find

$$p^\alpha = mA_\mu^\alpha \dot{X}^\mu, \quad (2 : \text{Extreme phase})$$

where upper case δX indicates an interval along a path of extreme phase, $\delta X^\mu = \delta x_{\text{extreme}}^\mu$. The path is a function of τ , written $X(\tau)$, and the dot indicates the derivative with respect to τ , $\dot{X}^\mu \equiv dX/d\tau$. The curves are defined in overlapping coordinate patches with A_μ^α effectively constant in each patch.

A second application of scattering transformations occurs with the particle paths. One way to move particles from one place to another is to translate them. If all particles and fields are translated by the same amount, no significant change occurs. But if we scatter translations, just one particle species field can be translated independent of the translations of other fields.

Translating the particle species field includes translating particle momentum. Commonly, translating a vector is thought to leave the vector unchanged. But this is not the general

case. From the study of first order relativistic wave equations like Dirac's only with different spin, the mathematics behind non-trivial translations of vectors is well known.[2, 3] One finds that to get momentum matrices that generate nontrivial translations of vectors, in general one must combine the vector with a second rank tensor $T_{(\Phi)}^{\gamma\delta}$ making a $4 + 16 = 20$ component quantity.[4, 5]

We assume the particle's momentum along each path of extreme phase is a parallel translated four-vector. In this way the change due to translation is acknowledged and canceled. Parallel translation of the momentum $p^\alpha(\tau)$ along a path $X(\tau)$ is shown in Sec. 3 to imply that

$$\dot{p}^\alpha = \eta_{\sigma\mu} T^{\alpha\sigma} \dot{X}^\mu, \quad (3 : \text{Parallel translation})$$

where $T^{\alpha\sigma}$ is a homogeneous function of $T_{(\Phi)}^{\gamma\delta}$.

The above displayed equations, i.e. mass, extreme phase and parallel translation, are the basis from which the rest follows. Except that it is at first convenient and later on necessary to restrict the tensor field A to the collection of all fields of local transformations from coordinates x to some local coordinates ξ ,

$$A_\mu^\alpha = \frac{\partial \xi^\alpha}{\partial x^\mu}. \quad (4 : \text{Local coordinates})$$

The first occurrence of local coordinates puts the quantum phase in the convenient form $\delta\Theta = p \cdot \delta\xi$. Near the end of the article A must be a field of transformations for the Christoffel connection to appear.

The first two equations displayed above lead to an arbitrary metric $g_{\mu\nu}$ dependent on the arbitrary tensor A . We call the combination (x, g) a 'curved spacetime' since in general g has non-vanishing curvature. The same coordinates x are used in the flat spacetime (x, η) and the curved spacetime (x, g) .

We show that τ is the proper time, the time for a particle at rest in the convenient local coordinates ξ . Having local coordinates ξ also implies that the metric g is locally Lorentzian, an important ingredient in producing general relativity. In this context the local transformation field, $A_\mu^\alpha = \partial \xi^\alpha / \partial x^\mu$, is known as a *tetrad* or *vierbein*.[6]

Next include parallel translation, the third displayed equation above. This produces a semi-classical equation of motion for the path $X(\tau)$,

$$m\ddot{X}^\mu = \left(A_\alpha^{-1\mu} \eta_{\sigma\nu} T^{\alpha\sigma} - m A_\beta^{-1\mu} \frac{\partial A_\nu^\beta}{\partial x^\lambda} \dot{X}^\lambda \right) \dot{X}^\nu, \quad (\text{Result : Equation of motion})$$

The equation is second order, with \ddot{X} a homogeneous quadratic function of the first derivatives \dot{X} . The equation of motion needs some interpretation.

The interpretation of the equation of motion begins by considering special cases. Let A be trivial, $A_\mu^\alpha = \delta_\mu^\alpha$. Then the derivative of $p \cdot p = -m^2$, (1) above, with respect to τ together with the equation of motion show that the tensor T must be antisymmetric, $T^{\alpha\sigma} = -T^{\sigma\alpha}$. In this case the equation of motion reduces to the equation for the path of a particle of charge q in an electromagnetic field F , where $m\ddot{X}^\alpha = \eta_{\sigma\mu}T^{\alpha\sigma}\dot{X}^\mu = \eta_{\sigma\mu}qF^{\alpha\sigma}\dot{X}^\mu$. We identify the electromagnetic field F by its antisymmetry and its relation to velocity \dot{X} and acceleration \ddot{X} .

As a second special case assume the electromagnetic field of the first case vanishes, i.e. $T^{\alpha\sigma} = qF^{\alpha\sigma} = 0$. Now, the equation of motion implies that the curve $X^\mu(\tau)$ is a geodesic of the metric $g_{\mu\nu}$ because the quantity $A_\beta^{-1}{}^\nu \partial A_\mu^\beta / \partial x^\lambda$ is shown to be the Christoffel connection C of $g_{\mu\nu}$. It is necessary to have A be a field of transformations to local coordinates ξ to show that the above quantity is the Christoffel connection C of the metric g .

In the local coordinate system and continuing with the case $T = 0$, the equation of motion is free fall, $\ddot{\Xi}^\alpha = 0$, where $\Xi(\tau)$ is the path of extreme phase $X(\tau)$ transformed to the ξ coordinate system. Thus we deduce one form of the Principle of Equivalence: In the absence of non-gravitational forces, i.e. $T = F = 0$, there exists a local inertial coordinate system ξ^α in which the particle falls freely. The particle moves along a straight line at constant speed in the free fall coordinate system.

Finally, as suggested by the special cases, we show that the equation of motion in general describes a charged massive particle in an electromagnetic field F that is related to T and the gravitational field of the metric g , which is a function of A . Thus we show that the equation of motion derived here is the equation for the motion of a massive, charged particle in general relativity,

$$m\ddot{X}^\nu = qg_{\sigma\mu}F^{\nu\sigma}\dot{X}^\mu - mC_{\lambda\mu}^\nu\dot{X}^\lambda\dot{X}^\mu. \quad (\text{Result : Equation of motion})$$

Comparing this way to other ways to deduce the motion of particles in electromagnetic and gravitational fields, the derivation here relies entirely on the principles of special relativity and quantum mechanics. We take from the literature the deduction of quantum fields from special relativity and quantum mechanics, in a process that we modify. While we include parallel translations, we do not pick and choose from the infinite variety of gauge groups; the transformations here are the rotations, boosts, and translations of the Poincaré group of symmetries in flat spacetime. We do not assume the Principle of Equivalence; we derive it. No Lagrangian, Hamiltonian, or action principles are introduced. Such considerations and methods with more rigor and less intuition than seeking paths of extreme phase may be discussed elsewhere.

2 Field and State

In this section, we derive some of the properties of the quantum field of a particle species. We follow Weinberg closely.[1] The differences with Weinberg involve the transformations of states and fields. First, we allow fields to be transformed by complete Poincaré transformations including translations, a generalization of convention. We do not agree that fields must be translation invariants.[7] Secondly, we transform the fields with a different Poincaré transformation than is applied to the states, not just a different rep, but a different transformation.

The quantum field $\psi_l(x)$ for a species of particles of mass m and spin j is constructed as a linear combination of annihilation and creation operators, $\psi_l(x) = \kappa\psi_l^+(x) + \mu\psi_l^-(x)$. One has an annihilation field ψ^+ and a creation field ψ^- given by

$$\begin{aligned}\psi_l^+(x) &= \sum_{\sigma} \int d^3p \, u_{l\sigma}(x, \vec{p}) a_{\sigma}(\vec{p}), \\ \psi_l^-(x) &= \sum_{\sigma} \int d^3p \, v_{l\sigma}(x, \vec{p}) a_{\sigma}^{\dagger}(\vec{p}),\end{aligned}\tag{2}$$

where $a_{\sigma}(\vec{p})$ and $a_{\sigma}^{\dagger}(\vec{p})$ are operators that remove or add an eigenstate of momentum \vec{p} and spin component σ . The time component of momentum, i.e. energy, is determined by the mass, $p^t = \sqrt{m^2 - \vec{p}^2}$. Since the fields are linear combinations of operators, quantum fields are also operators. We reserve the term ‘operator’ for the annihilation and creation operators a and a^{\dagger} .

The coefficient functions u and v can be determined from the ways quantities in (2) transform under Poincaré transformations to a new spacetime reference frame: (i) the operators a and a^{\dagger} transform with a unitary representation (rep), (ii) the coefficients u and v are required to be invariant and (iii) the quantum field transforms by a nonunitary rep.

We part company with Weinberg by scattering the spacetime transformations of fields and states. The field undergoes a Poincaré transformation in one reference frame while the states and operators are transformed in a different frame, the frames related by a Lorentz transformation λ . Call the coordinates x for the field frame and x_S for the state frame and let Λ be a Lorentz transformation applied to the fields and Λ_S be the transformation applied to the states. We have

$$x = \lambda x_S \quad \text{and} \quad \Lambda = \lambda \Lambda_S \lambda^{-1},\tag{3}$$

where S indicates ‘States’. The equivalence relation between Λ_S and Λ means that they are the same transformation in different frames. Upon transformation we have $x' = \Lambda x = \lambda x'_S$.

While the Lorentz transformations of fields and operators are equivalent, we allow different displacements, b and b_S . We assume the displacement for the operators and states

depends on the Lorentz transformation Λ , the event x , and the displacement of the field b .

$$b_S = b_S(\Lambda, x, b), \quad (4)$$

Then the field ψ is Poincaré transformed by (Λ, b) when the operators a and a^\dagger (and states) are transformed by (Λ_S, b_S) . Here, the notation (Λ, b) indicates a Lorentz transformation Λ (rotations, boosts) followed by a translation through a displacement b .

The operators transform just like single-particle states, so we often speak of transforming states rather than operators. In the Introduction and often elsewhere, we speak of transforming states. In practice we transform the annihilation and creation operators.

The transformation of the operators is unitary and is written $U(\Lambda_S, b_S)$. By the well-known transformation rule for operators, see [1] for details, we have

$$U(\Lambda_S, b_S)\psi_l^+(x)U^{-1}(\Lambda_S, b_S) = \sum_{\sigma} \int d^3p \, u_{l\sigma}(x, \vec{p}) e^{i\Lambda_S p \cdot b_S} \sqrt{\frac{(\Lambda_S p)^t}{p^t}} \sum_{\bar{\sigma}} D_{\sigma\bar{\sigma}}^{(j)}(W^{-1}) a_{\bar{\sigma}}(\vec{\Lambda_S p}), \quad (5)$$

where $p \cdot x \equiv \eta_{\alpha\beta} p^\alpha x^\beta$ is the spacetime scalar product, $D^{(j)}$ is a spin j unitary representation of rotations, $W(\Lambda_S, \vec{p})$ is the rotation, $k \rightarrow p \rightarrow \Lambda_S p \rightarrow k$ and $k = \{0, 0, 0, m\}$. Note that by (ii) above, the coefficients u remain invariant.

The expression for ψ^- differs from the expression for ψ^+ only by v for u , $-i$ for i , and $D^{(j)*}$ for $D^{(j)}$. It makes no sense here to write expressions for both; henceforth we consider ψ^+ . The discussion is similar for ψ^- , except for some special considerations with $D^{(j)*}$, see [1] for details.

The unitary transformation $U(\Lambda_S, b_S)$ is required to have the effect of (iii) a nonunitary transformation on the fields. (Technically, the reps for rotations can be both finite dimensional and unitary, but the reps of boosts must be nonunitary when finite dimensional because boosts are not compact.[1],p. 231) One requires that

$$U(\Lambda_S, b_S)\psi_l^+(x)U^{-1}(\Lambda_S, b_S) = \sum_{\bar{l}} D_{l\bar{l}}^{-1}(\Lambda, b)\psi_{\bar{l}}^+(\Lambda x + b), \quad (6)$$

where $\Lambda x + b$ are the transformed coordinates of the event x and $D(\Lambda, b)$ is the nonunitary matrix representing the Poincaré transformation (Λ, b) . The matrix $D_{l\bar{l}}(\Lambda, b)$ corresponds to the spin of the field ψ .

The translation part of $D_{l\bar{l}}(\Lambda, b)$ is trivial unless the spin of the field ψ is reducible and its spin composition contain ‘linked’ reps. In standard notation, the spin is $(A, B) \oplus (C, D) \oplus \dots$, where A, B, C, D are integers or half integers. For example, (A, B) and (C, D) are linked when $(C, D) = (A \pm 1/2, B \pm 1/2)$. The Dirac spinor, spin $(1/2, 0) \oplus (0, 1/2)$, has linked reps.

To find ψ^+ , (2), we consider the operators as known and solve (2), (5), (6) for $u(x, \vec{p})$. By (2), we can write (6) as a sum and integral over operators, transform the dummy variables \vec{p} to $\vec{\Lambda_S p}$, and equate integrands. We get

$$e^{i(\Lambda_S p) \cdot b_S} \sqrt{\frac{p^t}{(\Lambda_S p)^t}} D_{\bar{l}\bar{l}}(\Lambda, b) u_{\bar{l}\sigma}(x, \vec{p}) = u_{\bar{l}\sigma}(\Lambda x + b, \vec{\Lambda_S p}) D_{\bar{\sigma}\sigma}^j(W(\Lambda_S, \vec{p})), \quad (7)$$

where the D^j and D have changed sides. In this equation with $\Lambda = \mathbf{1}$ and $b = -x$, we have $\Lambda x + b = 0$. Then we can put $\Lambda = \tilde{\Lambda}$, with $x = \tilde{\Lambda}\tilde{x} + \tilde{b}$ and $b = -\tilde{\Lambda}\tilde{x} - \tilde{b}$, so that again we have $\Lambda x + b = 0$. Drop the tildes. Substituting both of these back in (7), we get

$$e^{+i\Lambda p \cdot [b_S(\Lambda, x, b) - \Lambda b_S(\mathbf{1}, x, -x) + b_S(\mathbf{1}, \Lambda x + b, -\Lambda x - b)]} \sum_{\bar{l}} D_{\bar{l}\bar{l}}(\Lambda, 0) u_{\bar{l}\sigma}(0, \vec{p}) = \sqrt{\frac{(\Lambda p)^t}{p^t}} \sum_{\bar{\sigma}} u_{\bar{l}\sigma}(0, \vec{\Lambda p}) D_{\bar{\sigma}\sigma}^{(j)}(W(\Lambda_S, p)). \quad (8)$$

The exponent on the left shows the derivation history with $\Lambda = \mathbf{1}$ and $b = -x$, etc. as the parameters in the displacement $b_S(\Lambda, x, b)$.

We have written (8) so that the right-hand-side does not depend on the coordinates x . Nor does it depend on the displacement b .

Since x and b occur only on the left in (8) and there only in the displacement function b_S , one seeks a suitable function $b_S(\Lambda, x, b)$ such that the expression in brackets in the phase on the left in (8) doesn't depend on x or b . One can show that $b_S(\Lambda, x, b)$ must be in the following form,

$$b_S^\mu(\Lambda, x, b) = b_S^\mu(\Lambda) - \Lambda_\sigma^\mu [A(x)]_\nu^\sigma x^\nu + [A(\Lambda x + b)]_\nu^\mu (\Lambda x + b)^\nu, \quad (9)$$

where $A(x)$ is an arbitrary second rank tensor field. For simplicity, we drop the x - and b -independent displacement, $b_S^\mu(\Lambda) = 0$. Note that one recovers $b_S = b$ when the field A is the identity, $A_\nu^\mu = \delta_\nu^\mu$, where δ is one for equal indices and zero otherwise. Also, b_S is not local; it depends on the values of the field A at both x and $\Lambda x + b$.

By (7), (8), (9), one finds expressions for the coefficients u and v in terms of the coefficients at the origin with the momentum of a particle at rest,

$$u_{l\sigma}(x, \vec{p}) = \sqrt{\frac{m}{p^t}} e^{ip \cdot Ax} \sum_{\bar{l}} D_{\bar{l}\bar{l}}(L, x) u_{\bar{l}\sigma}(0, \vec{0}). \quad (10)$$

To get to the rest frame momentum $\vec{p} = \vec{0}$, i.e. $p = k = \{0, 0, 0, m\}$, we considered (7) with $p = k$ and with Λ a transformation taking k to p' so that $W(\Lambda_S, \vec{p}) = 1$. Drop the primes. For details see [1], p. 195.

The expression (10) and the similar one for $v_{l\sigma}(x, \vec{p})$ along with Eq. (2) for ψ^+ and ψ^- determine many fundamental aspects of the quantum field $\psi_l(x) = \kappa\psi_l^+(x) + \mu\psi_l^-(x)$. Further developments are not needed here; see Ref. [1].

We take the phases $\pm p \cdot Ax$ of the coefficients $u_{l\sigma}(x, \vec{p})$ and $v_{l\sigma}(x, \vec{p})$ to be the phases of a particle and its antiparticle when it has momentum p . Let us consider just the phase with a positive sign.

The tensor field A is shown below to be related to the gravitational field. Many tests of quantum field theory occur in a slowly varying gravitational field. And in the tested calculations, the phases of the quantum fields are $p \cdot x$, not $p \cdot Ax$.

We can get local coordinates ξ such that the phase $p \cdot Ax = p \cdot \xi$, if A is the transformation

$$A_\mu^\alpha = \frac{\partial \xi^\alpha}{\partial x^\mu}. \quad (\text{Local coordinates}) \quad (11)$$

Not all tensor fields allow such an interpretation. The second order partials must commute, $\partial^2 \xi^\alpha / \partial x^\lambda \partial x^\mu = \partial^2 \xi^\alpha / \partial x^\mu \partial x^\lambda$, or

$$\frac{\partial A_\mu^\alpha}{\partial x^\lambda} = \frac{\partial A_\lambda^\alpha}{\partial x^\mu}. \quad (12)$$

These are the integrability conditions so that the field A can be a field of transformations.

Consider a neighborhood of the event x_0 , $x = x_0 + \delta x$. Then the neighborhood of x_0 is mapped into a neighborhood of $\xi_0 = Ax_0$, by

$$\xi^\alpha = A_\mu^\alpha x^\mu = A_\mu^\alpha x_0^\mu + A_\mu^\alpha \delta x^\mu = \xi_0^\alpha + \frac{\partial \xi^\alpha}{\partial x^\mu} \delta x^\mu = \xi_0^\alpha + \delta \xi^\alpha. \quad (13)$$

The existence of local coordinates is supported by the successes of quantum field theory.

Now we find timelike paths with extreme phase. Let us consider intervals δx so short that A varies negligibly along δx . Below we consider parallel translation of momentum, so p can change also. Assume p is effectively constant along δx . Then the change in the phase Θ associated with a particle of momentum p is

$$\delta \Theta = p \cdot A \delta x = \eta_{\alpha\beta} p^\alpha A_\mu^\beta \delta x^\mu$$

over such a displacement δx .

Let the extreme phase shift be $\delta \Theta_{\text{extreme}} = -m\delta\tau$, which defines τ . By calculus, one finds that the extreme phase shift occurs when $A\delta x$ is proportional to p ,

$$A_\mu^\alpha \delta x_{\text{extreme}}^\mu = A_\mu^\alpha \delta X^\mu = m^{-1} p^\alpha \delta \tau, \quad (\text{extreme } \delta \Theta) \quad (14)$$

where upper case X , as in δX , is used with displacements $\delta x_{\text{extreme}}$ that make $\delta \Theta$ extreme.

Continuing we can connect the δX with a second δX , perhaps for slightly different A and p . In this way we form a curve $X(\tau)$, with $A_\mu^\alpha(\tau)$ and $p^\alpha(\tau)$ the values of A and p along the curve $X(\tau)$.

Indicating the derivative with respect to τ with a dot,

$$\dot{X} \equiv dX/d\tau,$$

(14) becomes

$$p^\alpha = mA_\mu^\alpha \dot{X}^\mu. \quad (\text{Extreme phase}) \quad (15)$$

This equation relates the particle momentum p to the tangent \dot{X} of the path for extreme phase.

By (11), when A is a transformation to local coordinates ξ , the curve $X(\tau)$ transforms to a curve $\Xi(\tau)$ of extreme phase in coordinates ξ . It follows that $p^\alpha = m\dot{\Xi}$. Just as with x and X , the path of extreme phase in coordinates ξ lowercase is labeled with the uppercase $\Xi(\tau)$.

Next we consider parallel translation of the momentum which is why the momentum changes along the path $X(\tau)$.

3 Parallel translations

Translations of quantities with components, like wave functions ψ_j and four-vectors v^α , are generated by momentum matrices. Do not confuse momentum matrices which are generators with particle momenta which are eigenvalues of eigenstates. Momentum matrices are special cases of ‘vector matrices’, in particular momentum matrices commute to satisfy the Poincaré algebra. Famous vector matrices include the Dirac gamma matrices γ_{ij}^α that combined with a partial derivative and a four-spinor make up the Dirac equation, $\gamma_{ij}^\alpha \partial_\alpha \psi_j = -m\psi_i$. [8] The partial derivative transforms as a four-vector, spin $(1/2, 1/2)$, and the four-spinor transforms with spin $(1/2, 0) \oplus (0, 1/2)$, which is a reducible representation.

Since the Dirac equation first appeared, people have investigated first order relativistic wave equations with the same form as the Dirac equation. [2, 3] Among the results of this work, it is found that to translate a four-vector such as the particle momentum p^α , we must combine the particle momentum’s irreducible rep spin $(A, B) = (1/2, 1/2)$ with a choice of linked reps with spin $(C, D) = (1/2 \pm 1/2, 1/2 \pm 1/2) = (0, 0), (1, 0), (0, 1), (1, 1)$. These four choices are the spin composition for general 2nd order tensors. Therefore it suffices to combine the momentum p^α with some second order tensor $T_{(\Phi)}^{\gamma\delta}$ and get momentum matrices to generate translations of the $4 + 16 = 20$ component quantity Φ ,

$$\Phi = \begin{pmatrix} p^\alpha \\ T_{(\Phi)}^{\gamma\delta} \end{pmatrix}. \quad (16)$$

As part of the multicomponent object Φ , the four-vector p^α has the most general Poincaré transformations, including translations. Like A , we consider $T_{(\Phi)}$ to be a free parameter.

When needed for calculations we select a well known representation of the Poincaré algebra with angular momentum and boost generators given by [9]

$$(J^{\rho\sigma}) = -i \begin{pmatrix} \eta^{\sigma\mu}\delta_\nu^\rho - \eta^{\rho\mu}\delta_\nu^\sigma & 0 \\ 0 & +\eta^{\rho\gamma}\delta_\epsilon^\sigma\delta_\xi^\delta - \eta^{\sigma\gamma}\delta_\epsilon^\rho\delta_\xi^\delta + \eta^{\rho\delta}\delta_\xi^\sigma\delta_\epsilon^\gamma - \eta^{\sigma\delta}\delta_\xi^\rho\delta_\epsilon^\gamma \end{pmatrix}. \quad (17)$$

The momentum matrices of the rep, P^μ , are [4]

$$(P^\sigma) = \begin{pmatrix} 0 & P_{12}^\sigma \\ 0 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & \pi_1\delta_\gamma^\sigma\delta_\delta^\alpha + \pi_2\delta_\delta^\sigma\delta_\gamma^\alpha + \pi_3\eta^{\sigma\alpha}\eta_{\gamma\delta} + \pi_4\eta^{\sigma\rho}\eta^{\alpha\kappa}\epsilon_{\rho\kappa\gamma\delta} \\ 0 & 0 \end{pmatrix},$$

where ϵ is the antisymmetric symbol and the four constants π_i have dimensions of an inverse distance. There is another set of matrices that change $T_{(\Phi)}$ and leave p unchanged. We don't want that. The momentum matrices, displayed above with only the 12-block nonzero, change the four-vector momentum p^α and leave the tensor $T_{(\Phi)}$ unchanged.

Since the momentum matrices P^μ have nonzero components only in an off-diagonal block, the product of any two vanishes and the exponential of a linear combination is linear in the P^μ s. Then the translation matrix for a translation along a displacement δx is

$$\exp(-i\delta x_\sigma P^\sigma) = \mathbf{1} - i\delta x_\sigma P^\sigma,$$

where $\mathbf{1}$ is the 20×20 unit matrix. Clearly, the translation changes four-vector v^α to v'^α ,

$$v'^\alpha = v^\alpha + \eta_{\sigma\mu} T^{\alpha\sigma} \delta x^\mu, \quad (18)$$

where we abbreviate

$$T^{\alpha\sigma} \equiv -i(P_{12}^\sigma)_{\beta\gamma}^\alpha T_{(\Phi)}^{\beta\gamma}.$$

A large displacement can be constructed in many ways by various sequences of small displacements, so over finite intervals the translated four-vector v' may depend on path. Please note that parallel translation adds an inhomogeneous term $\eta T \delta x$ to v which for general T does not vanish when v vanishes.

The question arises: What four-vector at $x + \delta x$ is equivalent to the four-vector at x ? Is it the four-vector with the same components or the translated four-vector? We assume it is the translated four-vector, so that translation does not produce any innate change to the four-vector.

Parallel Translation of a Four-Vector. The translated four-vector v' is equivalent to the original four-vector v .

Let $v^\alpha(s)$ be a suitably differentiable four-vector function defined along the curve $x^\mu(s)$ where the real parameter s changes monotonically along the curve. We define the ‘translation covariant derivative’ D_T by its action on $v^\alpha(s)$,

$$\frac{D_T v^\alpha}{ds} \equiv \frac{dv^\alpha}{ds} - \eta_{\sigma\mu} T^{\alpha\sigma} \frac{dx^\mu}{ds}. \quad (19)$$

The change in the four-vector field from one event to a nearby event should not be determined by the difference in coordinates, i.e. the derivative, but by this covariant derivative that takes into account the changes brought about by parallel translation.

The next question is: What particle momentum at a nearby event is equivalent to the particle momentum at an original event? The answer is the same as above, but in this context it is the Dynamical Postulate.

Dynamical Postulate. A particle in a given eigenstate of momentum p remains in eigenstates of momenta equivalent to p as spacetime is translated.

As the quantum field ψ^+ is translated, i.e. the particle ‘moves’, its momenta change to equivalent momenta. The translation causes the interval $\delta X(\tau)$ obtained in (14) by extreme phase change, $\delta\Theta = p \cdot A\delta X$, to connect with the interval $\delta X'(\tau)$ obtained by a new extreme phase change, $\delta\Theta' = p' \cdot A\delta X'$, where the parallel momentum p' is given by (18).

Clearly, successive intervals create a path $X(\tau)$ such that the covariant derivative of the particle momentum vanishes along the path, $D_T p^\alpha/d\tau = 0$, where $p^\alpha(\tau)$ is the momentum at $X(\tau)$. We have, by (19),

$$\dot{p}^\alpha = \eta_{\sigma\mu} T^{\alpha\sigma} \dot{X}^\mu, \quad (\text{Parallel translation}) \quad (20)$$

where, as throughout, the dot stands for the derivative with respect to the parameter τ . This is the semi-classical equation of motion.

The equation of motion preserves the momentum along the path of extreme phase from event to event in spacetime. A ‘geodesic’ has a parallel transferred tangent, while the semi-classical motion (20) has a related but different notion, parallel translated momentum.

4 Curved Spacetime

Consider an event O and adjust the parameters τ of all paths of extreme phase $X(\tau)$ through O so that $X(0) = x_0$ are the coordinates of the event O . The collection of such paths forms the interior light cone at O , i.e. the set of all events that a particle at O could once have been at or can ever get to. We assume that pathological values of the arbitrary tensor field A are not allowed so that the interior light cone is well behaved. In general the paths $X(\tau)$ are curves, and in this section they find a natural place in a curved spacetime.

Much can be deduced from the mass equation, Eq. (1), i.e. $\eta_{\alpha\beta}p^\alpha p^\beta = -m^2$. By substituting (15), the extreme phase requirement along path $X(\tau)$, $p = mA\dot{X}$, one finds that

$$\frac{1}{m^2}\eta_{\alpha\beta}p^\alpha p^\beta = \eta_{\alpha\beta}A_\mu^\alpha A_\nu^\beta \dot{X}^\mu \dot{X}^\nu = -1,$$

or, by collecting some of the quantities together,

$$g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu = -1, \quad (21)$$

where the tensor field $g_{\mu\nu}(x)$ is defined in terms of the tensor field $A(x)$ by

$$g_{\mu\nu} \equiv \eta_{\alpha\beta}A_\mu^\alpha A_\nu^\beta \quad \text{and} \quad g^{\mu\nu} \equiv \eta^{\alpha\beta}A_\alpha^{-1\mu} A_\beta^{-1\nu}. \quad (22)$$

In this we assume A has an inverse, so that $g_{\mu\nu}$ has an inverse that we write with raised indices, $g^{\rho\sigma}$. Since the flat spacetime metric $\eta_{\alpha\beta}$ is symmetric, both $g_{\mu\nu}$ and $g^{\rho\sigma}$ are symmetric. We call $g_{\mu\nu}$ the ‘curved spacetime metric’, a term that will become more and more justified as we proceed.

If, as before in (11), we assume that the tensor field $A(x)$ is a field of transformations to local coordinates, $A_\mu^\alpha = \partial\xi^\alpha/\partial x^\mu$, then we have

$$g_{\mu\nu} = \eta_{\alpha\beta}A_\mu^\alpha A_\nu^\beta = \eta_{\alpha\beta} \frac{\partial\xi^\alpha}{\partial x^\mu} \frac{\partial\xi^\beta}{\partial x^\nu}.$$

By (21) and (22), one can quickly show that

$$\eta_{\alpha\beta}\dot{\Xi}^\alpha \dot{\Xi}^\beta = -1, \quad (23)$$

where $\Xi(\tau)$ is the path of extreme phase in coordinates ξ transformed from $X(\tau)$ by A .

Because the metric in (23) is the flat spacetime metric η , the transformation A gives locally flat spacetime coordinates ξ . It follows that $g_{\mu\nu}$ is locally Lorentzian. Being locally Lorentzian is an important property of the metric in general relativity.

Furthermore, we can transform the path $\Xi(\tau)$ at the event O at $\tau = 0$ to a frame so that $\dot{\Xi}(0)$ has only its time component nonzero, $\dot{\Xi}(0) = \{0, 0, 0, \dot{\Xi}_0^t\}$. Then, by (23), we have $\dot{\Xi}_0^t = 1$ and $d\Xi_0^t = d\tau$, so we are justified in calling the quantity τ *the ‘proper time’ along the path $X(\tau)$ of extreme phase*. The parameter τ is the time in a local Lorentz frame in which the particle is momentarily at rest.

In general relativity changing g is associated with transforming the coordinates of spacetime. In this paper the changes are mediated by the tensor field $A(x)$, which is an element of the quantum phase in (10). When A changes, $A \rightarrow A'$, the metric g changes due its definition: $g \equiv \eta AA$, indices suppressed, and $g' = \eta A'A'$. The coordinates of paths of extreme

phase $X(\tau)$ change because of the requirement $g\dot{X}\dot{X} = g'\dot{X}'\dot{X}' = -1$. Then the new tangent to a path of extreme phase is $\dot{X}' = A^{-1}\dot{X}$. When A changes, the metric g and the paths X change, but the coordinates x are unaffected.

Thus the word ‘event’ loses much of its intuitive meaning. When a change in A changes the path X , the particle may not pass through the same events. What happens at an event x depends on A .

Turn now to the particle momentum. Define the curved spacetime momentum $\bar{p}^\mu(\tau)$ along a path $X(\tau)$ to be

$$\bar{p}^\mu \equiv m\dot{X}^\mu,$$

so that $p^\alpha = A_\mu^\alpha \bar{p}^\mu$ by (15). By (21), one finds that

$$g_{\mu\nu}\bar{p}^\mu\bar{p}^\nu = -m^2. \quad (24)$$

Thus the mass m is the magnitude of the momentum $\bar{p}(\tau)$ calculated with the metric g .

Equation (20) mixes the flat spacetime particle momentum p^α and the curved spacetime quantity X^μ . [We identify p^α as ‘flat’ and X as ‘curved’ in part by $\eta_{\alpha\beta}p^\alpha p^\beta = -m^2$ and $g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu = -1$.] By writing p^α in terms of the curved spacetime momentum \bar{p}^μ and substituting in (20), we have an equation with the curved spacetime quantities \bar{p}^μ and X^μ . One finds that

$$\dot{\bar{p}}^\mu = m\ddot{X}^\mu = \left(A_\alpha^{-1\mu} \eta_{\sigma\nu} T^{\alpha\sigma} - mA_\beta^{-1\mu} \frac{\partial A_\nu^\beta}{\partial x^\lambda} \dot{X}^\lambda \right) \dot{X}^\nu, \quad (25)$$

where we have used (15) to write p in terms of \dot{X} . This is a second version of the semiclassical equation of motion (20).

5 Special Cases

The equation of motion (25) has two types of terms; we consider them one at a time. In Case 1, we simplify A , $A_\mu^\alpha = \delta_\mu^\alpha$. The equations of motion reduce to the Lorentz force law for a charged particle in an electromagnetic field. In Case 2 we simplify T , $T = 0$, and we show that the particle path is the motion of a particle in a gravitational field, provided that A is a set of transformations to local coordinates ξ . We put the two cases together in the following section.

Case 1. Let $A_\mu^\alpha(x) = \delta_\mu^\alpha$ and, in particular, $\partial A_\nu^\beta / \partial x^\lambda = 0$ everywhere.

The momentum $p^\alpha(\tau)$ at any proper time τ along a path $X(\tau)$ obeys the mass equation (1), $-m^2 = \eta_{\alpha\beta} p^\alpha p^\beta$. Take the derivative with respect to proper time. We get

$$0 = -\frac{dm^2}{d\tau} = 2\eta_{\alpha\beta} p^\alpha \dot{p}^\beta = 2m\eta_{\alpha\beta} A_\sigma^\alpha \dot{X}^\sigma \eta_{\lambda\rho} T^{\beta\lambda} \dot{X}^\rho,$$

where we write p^α in terms of \dot{X}^σ by (15) and we write \dot{p}^β in terms of \dot{X}^ρ by the version (20) of the semiclassical equation of motion. Simplifying using $A_\sigma^\alpha = \delta_\sigma^\alpha$, we find

$$0 = 2m\eta_{\alpha\beta} \dot{X}^\alpha \eta_{\lambda\rho} T^{\beta\lambda} \dot{X}^\rho = m \left(\eta_{\alpha\beta} \dot{X}^\alpha \right) \left(\eta_{\rho\lambda} \dot{X}^\rho \right) \left(T^{\beta\lambda} + T^{\lambda\beta} \right) = p_\beta p_\lambda \left(T^{\beta\lambda} + T^{\lambda\beta} \right),$$

by the symmetry evident in the p_α factors, where $p_\alpha \equiv m\eta_{\alpha\sigma} \dot{X}^\sigma$. There are enough possible momenta p that determine enough possible tangents \dot{X}^μ to deduce that $T^{\beta\lambda} + T^{\lambda\beta}$ must vanish in the well-behaved interior light cone of any event and hence everywhere. It follows that T must be antisymmetric everywhere,

$$T^{\beta\lambda} = -T^{\lambda\beta}. \quad (26)$$

Then the semi-classical equation of motion (20) is just the Lorentz force law,[10]

$$\dot{p}^\alpha = q\eta_{\sigma\mu} F^{\alpha\sigma} \dot{X}^\mu, \quad (27)$$

for a particle of charge q in an electromagnetic field F , with $T^{\alpha\sigma} = qF^{\alpha\sigma}$.

We identify (27) as the Lorentz force law by the placement of force \dot{p}^α , antisymmetric field $F^{\alpha\beta}$ and particle velocity \dot{X} . We do not discuss the determination of the electromagnetic field from charges and currents.[11]

Let us now consider the other type of terms in the equation of motion (25).

Case 2. Let $T^{\alpha\sigma} = 0$, so that the equation of motion (25) reduces to

$$\ddot{X}^\nu = -A_\beta^{-1}{}^\nu \frac{\partial A_\mu^\beta}{\partial x^\lambda} \dot{X}^\lambda \dot{X}^\mu = -\frac{1}{2} A_\beta^{-1}{}^\nu \left(\frac{\partial A_\mu^\beta}{\partial x^\lambda} + \frac{\partial A_\lambda^\beta}{\partial x^\mu} \right) \dot{X}^\lambda \dot{X}^\mu = -C_{\lambda\mu}^\nu \dot{X}^\lambda \dot{X}^\mu, \quad (28)$$

which defines C . We want to relate C to the Christoffel connection of the metric g .

The Christoffel symbol of the first kind is defined to be [12]

$$[\mu\nu, \rho] \equiv \frac{1}{2} \left(\frac{\partial g_{\mu\rho}}{\partial x^\nu} + \frac{\partial g_{\nu\rho}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right).$$

By the definition of the metric g in terms of the tensors A in (22), i.e. $g = \eta AA$, one can manipulate the Christoffel symbol of the first kind to show that

$$[\mu\nu, \rho] = g_{\rho\sigma} C_{\mu\nu}^{\sigma} + \frac{1}{2} \left[A_{\mu}^{\alpha} \left(\frac{\partial A_{\rho}^{\beta}}{\partial x^{\nu}} - \frac{\partial A_{\nu}^{\beta}}{\partial x^{\rho}} \right) + A_{\nu}^{\alpha} \left(\frac{\partial A_{\rho}^{\beta}}{\partial x^{\mu}} - \frac{\partial A_{\mu}^{\beta}}{\partial x^{\rho}} \right) \right].$$

Looking at this, we see that C is the Christoffel connection when the terms in parentheses vanish, i.e. when

$$\frac{\partial A_{\rho}^{\beta}}{\partial x^{\nu}} = \frac{\partial A_{\nu}^{\beta}}{\partial x^{\rho}}.$$

But this is just the integrability condition needed to find functions $\xi^{\beta}(x)$ so that

$$A_{\nu}^{\beta} = \frac{\partial \xi^{\beta}}{\partial x^{\nu}}.$$

We recognize the local coordinates assumption first introduced above in (11).

Thus, when A is a field of transformations to local coordinates, the quantity C is the Christoffel connection of the metric g , [12]

$$C_{\mu\nu}^{\sigma} = g^{\rho\sigma} [\mu\nu, \rho] = \frac{g^{\rho\sigma}}{2} \left(\frac{\partial g_{\mu\rho}}{\partial x^{\nu}} + \frac{\partial g_{\nu\rho}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} \right).$$

Then the equation of motion reduces to

$$\ddot{X}^{\nu} = -C_{\lambda\mu}^{\nu} \dot{X}^{\lambda} \dot{X}^{\mu},$$

which is the general relativistic equation for the path of a massive particle in the gravitational field associated with the metric g .

As in Case 1, we identify the equation as the equation of a particle in a gravitational field by the placement in the equation of the acceleration \ddot{X} , connection C and velocity \dot{X} . We do not discuss the relation of the gravitational field to its sources, i.e. the energy, momentum and angular momentum of matter.[11]

Multiplying the semiclassical equation of motion (28) in this case by A , i.e. $A\ddot{X} = -AA^{-1}(\partial A/\partial x)\dot{X}\dot{X}$, indices suppressed, put both terms on the right side, and write out the derivatives with respect to proper time τ (replace the dots), we get that

$$0 = A_{\nu}^{\alpha} \frac{d^2 X^{\nu}}{d^2 \tau} + \frac{\partial A_{\nu}^{\alpha}}{\partial x^{\lambda}} \frac{dX^{\lambda}}{d\tau} \frac{dX^{\nu}}{d\tau} = \frac{\partial \xi^{\alpha}}{\partial x^{\nu}} \frac{d^2 X^{\nu}}{d^2 \tau} + \left[\frac{d}{d\tau} \left(\frac{\partial \xi^{\alpha}}{\partial x^{\nu}} \right) \right] \frac{dX^{\nu}}{d\tau} = \frac{d}{d\tau} \left(\frac{\partial \xi^{\alpha}}{\partial x^{\nu}} \frac{dX^{\nu}}{d\tau} \right) = \frac{d^2 \Xi^{\alpha}}{d\tau^2}$$

or, with dots,

$$\ddot{\Xi}^{\alpha} = 0, \tag{29}$$

for the path of extreme phase $\Xi(\tau)$ transformed by A from $X(\tau)$. This is the equation for force free, acceleration free motion, i.e. free fall. Free fall in the local system of coordinates ξ occurs here with $T^{\alpha\sigma} = 0$, which in view of Case 1, means that gravity is the only force acting on the particle.

Therefore, when A is a field of local coordinate transformations, we have derived the *Principle of Equivalence*, which states that *under the influence of purely gravitational forces there exists a freely falling coordinate system ξ^α in which the equation for the path $\Xi^\alpha(\tau)$ of the particle is the equation for a straight line, $\ddot{\Xi}^\alpha = 0$.* [13]

6 Combined Electromagnetic/Gravitational Motion

In the special cases of the last section, the T term on the right in (25) is the electromagnetic Lorentz force term and the term with the A s is gravitational. Now, in the general case, let us see how to combine the two cases.

Let us assume in this section that A is the set of local transformations $A_\mu^\alpha = \partial\xi^\alpha/\partial x^\mu$ from X to local coordinates ξ ; we assume that (12) holds.

The A terms are fine as is. The collection of A s and their derivatives that define C in (28) do not depend on T and do not change simply because T no longer vanishes. The steps in Case 2 above still show that C is the Christoffel connection of g .

The T term in the semiclassical equation of motion (25) needs attention. We want to have the following form for the equation of motion (25),

$$m\ddot{X}^\nu = qg_{\sigma\mu}F^{\nu\sigma}\dot{X}^\mu - mC_{\lambda\mu}^\nu\dot{X}^\lambda\dot{X}^\mu, \quad (30)$$

where we have substituted

$$A_\alpha^{-1}{}^\nu\eta_{\sigma\mu}T^{\alpha\sigma} = qg_{\sigma\mu}F^{\nu\sigma}, \quad (31)$$

$q \neq 0$, and where F is now a free tensor field.

We ask: Must F be antisymmetric here as it was in Case 1 above? As in Case 1, antisymmetry and the proper placement of acceleration and velocity in the equation of motion are the only requirements needed for us to identify F as the electromagnetic field.

Since C is the same as it was in Case 2, the work leading to (29) can be adapted here. We multiply (30) by A_ν^β and note that the quantity $mA\ddot{X} + mAC\dot{X}\dot{X}$ equals $m\ddot{\Xi}$. We get

$$m\ddot{\Xi}^\alpha = A_\nu^\alpha qg_{\sigma\mu}F^{\nu\sigma}\dot{X}^\mu.$$

Next, take the derivative of (23) with respect to proper time τ , i.e. $d(\eta\dot{\Xi}\dot{\Xi})/d\tau = d(-1)/d\tau = 0$, and we find that

$$0 = 2\eta_{\alpha\beta}\dot{\Xi}^\alpha\ddot{\Xi}^\beta = 2\eta_{\alpha\beta}\left(A_\rho^\alpha\dot{X}^\rho\right)\left(A_\nu^\beta qg_{\sigma\mu}F^{\nu\sigma}\dot{X}^\mu\right).$$

Now make ηAA into a g , and we get

$$0 = 2q \left(g_{\rho\nu} \dot{X}^\rho \right) \left(g_{\sigma\mu} \dot{X}^\mu \right) F^{\nu\sigma} = 2\bar{p}_\nu \bar{p}_\sigma F^{\nu\sigma} = \bar{p}_\nu \bar{p}_\sigma (F^{\nu\sigma} + F^{\sigma\nu}) ,$$

where $\bar{p}_\mu \equiv g_{\mu\nu} \bar{p}^\nu = g_{\mu\nu} m \dot{X}^\nu$. There are enough possible particle momenta \bar{p} to determine that $F^{\nu\sigma} + F^{\sigma\nu} = 0$. Thus we have shown that F is antisymmetric,

$$F^{\nu\sigma} = -F^{\sigma\nu} . \quad (32)$$

Since F must be antisymmetric, we identify it as the electromagnetic field.

Thus the semiclassical equation of motion (25) is the equation (30) for the classical path of a particle of mass m and charge q in an electromagnetic field $F^{\nu\sigma}$ and in the gravitational field due to the metric $g_{\mu\nu}$. The electromagnetic field $F^{\nu\sigma}$ is related to the tensor field T by (31). The tensor field T is part of the parallel translation rule for four-vectors in (18). And the metric $g_{\mu\nu}$ is related by (22) to the tensor field A which is part of the quantum phase found in (10).

We can now follow standard treatises[14] and introduce the ‘covariant derivative’ of \dot{X} as

$$\frac{D\dot{X}^\nu}{d\tau} \equiv \ddot{X}^\nu + C_{\lambda\mu}^\nu \dot{X}^\lambda \dot{X}^\mu . \quad (33)$$

Then (30) can be written in terms of ‘covariant’ quantities. One has

$$\frac{D\dot{X}^\nu}{d\tau} = \frac{q}{m} g_{\sigma\mu} F^{\nu\sigma} \dot{X}^\mu . \quad (34)$$

This equation is well known to be covariant, having the same same form when the metric g is transformed to some other metric g' . [6] And transforming $g_{\mu\nu} = \eta_{\alpha\beta} A_\mu^\alpha A_\nu^\beta$, entails replacing the field A . The freedom to choose A , see (9), arises from the arbitrariness inherent when allowing the operators and states to be translated through one displacement b_S while the field is translated through a possibly different displacement b . General covariance is thus grounded in the scattering of displacements among the fields and states of quantum theory.

The work here is founded on special relativity and quantum mechanics. It is remarkable that the derivation of aspects of quantum field theory can be adjusted to obtain the semiclassical equations of motion for a massive, charged particle in electromagnetic and gravitational fields. It is widely accepted that electromagnetic and gravitational forces are the only two long range forces. And it is just these two forces that this process produces.

References

- [1] Weinberg, S., *The Quantum Theory of Fields*, Vol. I (Cambridge University Press, Cambridge, 1995), Chapter 5.
- [2] For a fairly early work, see Bhabha, H. J., *Rev. Mod. Phys.* **17**, 200 (1945). Vector matrices in terms of dotted, undotted spinors.
- [3] See, for example, Lyubarskii, G. Ya., *The Applications of Group Theory in Physics*, translated by S. Dedijs (Pergamon Press, New York, 1960) and references therein. Derives vector matrices in terms of Clebsch-Gordon coefficients.
- [4] R. Shurtleff, on-line article, arXiv:gr-qc/0502021, 2005.
- [5] R. Shurtleff, on-line article, arXiv:math-ph/0401002v3, 2008.
- [6] See, for example, Misner, C. W., Thorne, K. S., Wheeler, J. A., *Gravitation*, (W. H. Freeman, San Francisco, 1973).
- [7] See, for example, Raymond, P., *Field Theory*, (Benjamin/Cummings, London, 1981), p. 19 line 17.
- [8] See, for example, Messiah, A., *Quantum Mechanics*, Vol. II, translated by J. Potter, (North-Holland, Amsterdam, 1966), Chapter XX.
- [9] See, for example, Weinberg, S., *Gravitation and Cosmology*, (John Wiley & Sons, New York, 1972), Chap. 2, Sec. 12, p. 59.
- [10] See, for example, Jackson, J. D., *Classical Electrodynamics*, (John Wiley & Sons, New York, 1962), p. 405
- [11] Maxwell's equations with sources from an earlier paper with this point of view: R. Shurtleff, on-line article, arXiv:hep-th/0406249v2 , 2004.
- [12] See, for example, Adler, R., Bazin, M., Schiffer, M., *An Introduction to General Relativity*, (McGraw-Hill, New York, 1965).
- [13] See, for example, Weinberg, S., *Gravitation and Cosmology*, (John Wiley & Sons, New York, 1972), Chap. 3, Sec. 1, p. 70.
- [14] See, for example, Dirac, P. A. M., *General Theory of Relativity*, (Princeton University Press, 1996), (originally published: J. Wiley, New York, 1975), Chapters 7 - 10.